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Convergence analysis of waveform relaxation methods for neutral differential-functional systems[☆]

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Abstract

In this paper, the problems of convergence and superlinear convergence of continuous-time waveform relaxation method applied to Volterra type systems of neutral functional-differential equations are discussed. Under a Lipschitz condition with time- and delay-dependent right-hand side imposed on the so-called splitting function, more suitable conditions about convergence and superlinear convergence of continuous-time WR method are obtained. We also investigate the initial interval acceleration strategy for the practical implementation of the continuous-time waveform relaxation method, i.e., discrete-time waveform relaxation method. It is shown by numerical results that this strategy is efficacious and has the essential acceleration effect for the whole computation process.

MSC: 65L05

Keywords: Neutral differential equations; Convergence; Time-dependent Lipschitz condition; Waveform relaxation; Acceleration

1. Introduction

Waveform relaxation (WR) is an iterative method for very large systems of ordinary differential equations (ODEs). It differs from classical iterative methods in that, it iterates with functions in a function space (continuous-time) instead of with finite sets of discrete variables. Such a technique was first proposed in [16] for time domain analysis of large differential systems modelling electrical networks. This method was further studied in [17–22] and many other authors for ODEs or PDEs (see, for example, [2,3,8,9,12,15,23,24]).

As WR method applied to delay differential equations, we can refer to [4,5,26] and the references therein, where functional-differential systems of neutral type are not discussed. We can also refer to the articles [1,7,10,13,14] and the references therein for differential-algebraic systems.

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Recently, articles [6,11] give some consideration of WR methods for Volterra functional-differential systems of the form

$$\begin{cases} y'(t) = f(t, y(\cdot), y'(\cdot)), & t \in I = [0, T], \\ y(t) = g(t), & t \in I_\tau = [-\tau, 0], \end{cases} \tag{1.1}$$

where the function $f : I \times C_g(I, R^n) \times C_{g'}(I, R^n) \to R^n$ and $g : [-\tau, 0] \to R^n$ is a given initial function, which is continuous with its first-order derivative and satisfies the consistency condition $g'(0) = f(t, y(\cdot), y'(\cdot))|_{t=0}$. Here, we denote by $C_g(I, R^n)$ the class of continuous functions defined on I with values in R^n which are equal to g for $t \in [-\tau, 0]$. Similarly, $C_{g'}(I, R^n)$ stands for the class of piecewise continuous functions defined on I with values in R^n which are equal to g' for $t \in [-\tau, 0]$.

It is known that many systems can be reduced to problems of the form (1.1). Such as

(1) $f(t, y(\cdot), y'(\cdot)) = \Phi(t, y(\alpha_1(t)), \ldots, y(\alpha_p(t)), y'(\beta_1(t)), \ldots, y'(\beta_q(t))), -\tau \le \alpha_i(t), \beta_j(t) \le t, i = 1, \ldots, p, j = 1, \ldots, q,$
(2) $f(t, y(\cdot), y'(\cdot)) = \Phi(t, y(\alpha_0(t)), y'(\beta_0(t)), \int_0^{\alpha_2(t)} K(t, s, y(\alpha_1(s)), y(\alpha_2(s)), y'(\beta_1(s)), y'(\beta_2(s)))\mathrm{d}s), -\tau \le \alpha_i(t), \beta_i(t) \le t, i = 0, 1, 2,$
(3) $f(t, y(\cdot), y'(\cdot)) = \Phi(t, y(\alpha_0(t)), \max_{-\tau \le s \le \alpha_1(t)} y(s), y'(\beta_0(t)), \max_{-\tau \le s \le \beta_1(t)} y'(s)), -\tau \le \alpha_i(t), \beta_i(t) \le t, i = 0, 1.$

and many others.

The WR method considered in these two articles is as follows

$$\begin{cases} y'_{k+1}(t) = F(t, y_{k+1}(\cdot), y_k(\cdot), y'_{k+1}(\cdot), y'_k(\cdot)), & t \in I, \\ y_{k+1}(t) = g(t), & t \in [-\tau, 0], \end{cases} \tag{1.2}$$

where $k = 0, 1, \ldots$ and $y_0(t)$ is a given starting function which satisfies the initial condition $y_0(t) = g(t)$ for $t \in [-\tau, 0]$. The function F, which is called the splitting function, is chosen to attempt to decouple systems (1.1) into easily solvable independent subsystems, which may then be solved separately. We see that the WR method (1.2) is very general.

The splitting function F is minimally assumed to satisfy a consistency condition, which ensures that the solution to (1.1) is a fixed one of (1.2), i.e.,

$$F(t, p(\cdot), p(\cdot), p'(\cdot), p'(\cdot)) = f(t, p(\cdot), p'(\cdot)) \tag{1.3}$$

for any function $p \in C_g(I, R^n)$.

It is valuable to review some prominent results given by those two articles.

In [11], the splitting function F is assumed to satisfy the following Lipschitz condition

$$\begin{aligned} &\|F(t, u(\cdot), v(\cdot), w(\cdot), x(\cdot)) - F(t, \bar{u}(\cdot), \bar{v}(\cdot), \bar{w}(\cdot), \bar{x}(\cdot))\| \\ &\quad \le L_1\|u - \bar{u}\|_t + L_2\|v - \bar{v}\|_t + K_1\|w - \bar{w}\|_t + K_2\|x - \bar{x}\|_t, \end{aligned} \tag{1.4}$$

where L_1, L_2, K_1 and K_2 are nonnegative constants. Under this condition, if $0 < K = \frac{K_2}{1-K_1} < 1$ (i.e., $K_1 + K_2 < 1$), the WR method (1.2) is convergent and the convergence of y_k to y is only linear and the rate of convergence is K, where y is the solution of system (1.1).

Later, in [6] F is assumed to satisfy

$$\begin{aligned} &|||F(t, u(\cdot), v(\cdot), w(\cdot), x(\cdot)) - F(t, \bar{u}(\cdot), \bar{v}(\cdot), \bar{w}(\cdot), \bar{x}(\cdot))||| \\ &\quad \le L_1|||u - \bar{u}|||_t + L_2|||v - \bar{v}|||_{\alpha(t)} + K_1|||w - \bar{w}|||_t + K_2|||x - \bar{x}|||_{\beta(t)}, \end{aligned} \tag{1.5}$$

where $|||\cdot||| : R^n \to R_+^{n'} (R_+ = [0, +\infty))$ denotes a vector norm in the space R^n, L_1, L_2, K_1, K_2 are nonnegative square matrices of dimension n' and $\alpha(t) \in C^1(I, I)$, $\beta(t) \in C(I, I)$ are nondecreasing functions which satisfy $0 \le \alpha(t), \beta(t) \le t$ for $t \in I$. Under this delay-dependent Lipschitz condition, more suitable conditions for convergence of (1.2) and delay-dependent error estimates are obtained. For completeness, we list these convergence conditions as follows.

(1) If $\rho(K_1) < 1$ and matrix $K = (1 - K_1)^{-1}K_2$ is irreducible with $\rho(K) < 1$, the sequence y_k defined by (1.2) converges uniformly in I to y;

(2) If there exists some $\delta > 0$ such that the function $\beta(t)$ in (1.5) satisfies $\beta(t) = 0$ for $t \in [0, \delta]$ and $\inf_{t\in[\delta,T]}(t - \beta(t)) = \delta$, then one will obtain the solution of (1.1) after a finite number of steps of the WR method (1.2).

Since the other convergence conditions given in [6] depend on the initial approximation y_0, we omit them at this moment.

Applying the convergence conditions given in [6,11] to the WR method

$$\begin{cases} y'_{k+1}(t) = y_{k+1}(t) + A(t)y_k\left(\frac{t}{2}\right) + B(t)y'_k\left(\frac{t}{2}\right), & t \in [0, T], \\ y_{k+1}(0) = 0, & t = 0, \end{cases} \tag{1.6}$$

$k = 0, 1, 2, \ldots$, we arrive at $\max_{0\le t\le T} \|B(t)\| < 1$ (or $\max_{0\le t\le T} \rho(|||B(t)|||) < 1$) to guarantee the convergence of y_k to y if we do not impose any restriction on the initial approximation y_0 except $y_0(0) = 0$.

In the present paper, we also consider the problems of convergence and superlinear convergence of the WR method (1.2) to systems (1.1) under a time- and delay-dependent Lipschitz condition of the splitting function F (see Condition 1 in Section 2). It is shown that, if the nonnegative function $\beta(t)$ mentioned above satisfies $\beta(t) < t$ for $t \in (0, T]$, the condition $K_1(0) + K_2(0) < 1$ is sufficient to guarantee the convergence of the WR method (1.2). Moreover, if $K_2(0) = 0$ we can obtain the superlinear convergence of y_k to y. Therefore, for the WR method (1.6) we can obtain the convergence and superlinear convergence of y_k to y under the condition $\|B(0)\| < 1$ and $\|B(0)\| = 0$, respectively.

We also consider the case $\beta(t) = t$ on some time points $t = t_i$, $0 \le i \le p$. It is shown that the WR method (1.2) is convergent in each subinterval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, p-1$ provided that $K_1(t_i) + K_2(t_i) < 1$ holds for $0 \le i \le p$.

The organization of this paper is as follows. Sharper error estimates and more suitable convergence conditions are obtained in Section 2. In Section 3, we consider two special cases to illustrate the superlinear convergence of the WR method (1.2). We present our numerical experiments in Section 4, and we will see that the numerical results are identical with our theoretical conclusions.

2. Convergence analysis

Assume that the splitting function F given in (1.2) satisfies the following time-dependent Lipschitz condition.

Condition 1. There exist nonnegative functions $\bar{L}_1(t)$, $\bar{L}_2(t)$, $K_1(t)$, $K_2(t) \in C(I, R^+ \cup \{0\})$ and nondecreasing functions $\alpha(t) \in C^1(I, I)$, $\beta(t) \in C(I, I)$ satisfying $\alpha(t) \le t$ for $t \in I$, $\beta(t) < t$ for $t \in (0, T]$ and $K_1(t) < 1$ for $t \in I$ such that for any functions $u(t), v(t), \bar{u}(t), \bar{v}(t) \in C_g(I, R^n)$, $x(t), w(t), \bar{x}(t), \bar{w}(t) \in C_{g'}(I, R^n)$, the splitting function F satisfies

$$\begin{aligned} &\|F(t, u(\cdot), v(\cdot), w(\cdot), x(\cdot)) - F(t, \bar{u}(\cdot), \bar{v}(\cdot), \bar{w}(\cdot), \bar{x}(\cdot))\| \\ &\quad \le \bar{L}_1(t)\|u - \bar{u}\|_t + \bar{L}_2(t)\|v - \bar{v}\|_{\alpha(t)} + K_1(t)\|w - \bar{w}\|_t + K_2(t)\|x - \bar{x}\|_{\beta(t)}. \end{aligned} \tag{2.1}$$

Here and below, for any function $v(t)$, $\|v\|_t = \sup_{0\le s\le t} \|v(s)\|$ and $\|\cdot\|$ stands for some given norm.

By Condition 1, it is easy to get

$$\alpha(0) = 0, \qquad \beta(0) = 0. \tag{2.2}$$

Following the approach in [6], we set $z(t) = y'(t)$ to obtain

$$y(t) = \begin{cases} g(0) + \int_0^t z(s)\mathrm{d}s, & t \in [0, T], \\ g(t), & t \in [-\tau, 0], \end{cases} \tag{2.3}$$

which we write in short as

$$y(t) = J(z(t)), \quad t \in [0, T]. \tag{2.4}$$

Now, instead of considering the WR method (1.2), we consider the following WR iteration scheme

$$z_{k+1}(t) = \begin{cases} F(t, J(z_{k+1}(\cdot)), J(z_k(\cdot)), z_{k+1}(\cdot), z_k(\cdot)), & t \in [0, T], \\ g'(t), & t \in [-\tau, 0]. \end{cases} \tag{2.5}$$

In the following of this paper, we will consider the convergence of the sequence z_k generated by the WR method (2.5) to z. It is obvious that such convergence is equivalent to the convergence of the sequence y_k generated by the WR method (1.2) to y.

Define

$$\bar{K}_1(t) = \max_{0\le s\le t} K_1(s), \qquad \bar{K}_2(t) = \max_{0\le s\le t} K_2(s), \qquad \bar{L}_1 = \max_{0\le t\le T} \bar{L}_1(t), \qquad \bar{L}_2 = \max_{0\le t\le T} \bar{L}_2(t). \tag{2.6}$$

For $t \in I$, it is obvious that $K_1(t) \le \bar{K}_1(t)$, $K_2(t) \le \bar{K}_2(t)$, the functions $\bar{K}_1(t)$ and $\bar{K}_2(t)$ are continuous and nondecreasing and $\bar{K}_1(t) < 1$.

Define

$$e_k(t) = \|z_k - z\|_t, \quad k = 0, 1, \ldots. \tag{2.7}$$

By (2.5) and Condition 1, we have

$$\begin{aligned} \|z_{k+1}(t) - z(t)\| &= \|F(t, J(z_{k+1}(\cdot)), J(z_k(\cdot)), z_{k+1}(\cdot), z_k(\cdot)) - F(t, J(z(\cdot)), J(z(\cdot)), z(\cdot), z(\cdot))\| \\ &\le \bar{L}_1(t) \sup_{0\le s\le t} \|J(z_{k+1}(s)) - J(z(s))\| + \bar{L}_2(t) \sup_{0\le s\le \alpha(t)} \|J(z_k(s)) - J(z(s))\| \\ &\quad + K_1(t) \sup_{0\le s\le t} \|z_{k+1}(s) - z(s)\| + K_2(t) \sup_{0\le s\le \beta(t)} \|z_k(s) - z(s)\| \\ &\le \bar{L}_1 \int_0^t e_{k+1}(s)\mathrm{d}s + \bar{L}_2 \int_0^{\alpha(t)} e_k(s)\mathrm{d}s + \bar{K}_1(t)e_{k+1}(t) + \bar{K}_2(t)e_k(\beta(t)), \end{aligned}$$

and this implies

$$e_{k+1}(t) \le \bar{L}_1 \int_0^t e_{k+1}(s)\mathrm{d}s + \bar{L}_2 \int_0^{\alpha(t)} e_k(s)\mathrm{d}s + \bar{K}_1(t)e_{k+1}(t) + \bar{K}_2(t)e_k(\beta(t)),$$

i.e.,

$$e_{k+1}(t) \le L_1(t) \int_0^t e_{k+1}(s)\mathrm{d}s + L_2(t) \int_0^{\alpha(t)} e_k(s)\mathrm{d}s + K(t)e_k(\beta(t)), \tag{2.8}$$

where

$$L_1(t) = \frac{\bar{L}_1}{1 - \bar{K}_1(t)}, \qquad L_2(t) = \frac{\bar{L}_2}{1 - \bar{K}_1(t)}, \qquad K(t) = \max_{0\le s\le t} \frac{\bar{K}_2(s)}{1 - \bar{K}_1(s)}. \tag{2.9}$$

It is also obvious that the nonnegative functions $L_1(t)$, $L_2(t)$ and $K(t)$ are continuous and nondecreasing with

$$K(0) = \frac{\bar{K}_2(0)}{1 - \bar{K}_1(0)} = \frac{K_2(0)}{1 - K_1(0)}. \tag{2.10}$$

With inequality (2.8), we have

Lemma 2.1. *The function $e_{k+1}(t)$ defined in (2.7) satisfies*

$$e_{k+1}(t) \le L_1(t)\mathrm{e}^{L(t)} \int_0^t \mathrm{e}^{-L(s)} u_k(s)\mathrm{d}s + u_k(t),$$

where $L(t) = \int_0^t L_1(s)\mathrm{d}s$ and $u_k(t) = L_2(t) \int_0^{\alpha(t)} e_k(s)\mathrm{d}s + K(t)e_k(\beta(t))$.

Proof. Set $w_{k+1}(t) = \int_0^t e_{k+1}(s)\mathrm{d}s$. Then by (2.8) we have

$$w'_{k+1}(t) \le L_1(t)w_{k+1}(t) + u_k(t), \qquad w_{k+1}(0) = 0.$$

Hence, due to $L_1(t) \geq 0$ and the differential inequalities (see [25]) it follows that $w_{k+1}(t) \leq w(t)$ holds for $t \in I$, where $w(t) = \int_0^t \mathrm{e}^{L(t)-L(s)} u_k(s)\mathrm{d}s$ is the solution of the system

$$w'(t) = L_1(t)w(t) + u_k(t), \qquad w(0) = 0.$$

Then, we arrive at

$$e_{k+1}(t) = w'_{k+1}(t) \leq L_1(t)w_{k+1}(t) + u_k(t) \leq L_1(t)w(t) + u_k(t),$$

i.e., $e_{k+1}(t) \leq L_1(t)\mathrm{e}^{L(t)} \int_0^t \mathrm{e}^{-L(s)} u_k(s)\mathrm{d}s + u_k(t)$ for $t \in I$. $\square$

From Lemma 2.1, we have

$$\begin{aligned} e_{k+1}(t) \leq{}& L_1(t)\mathrm{e}^{L(t)} \int_0^t \left\{ \left(-\frac{L_2(s)}{L_1(s)} \right) \int_0^{\alpha(s)} e_k(z)\mathrm{d}z \right\} \mathrm{d}\mathrm{e}^{-L(s)} \\ &+ L_1(t)\mathrm{e}^{L(t)} \int_0^t \mathrm{e}^{-L(s)} K(s)e_k(\beta(s))\mathrm{d}s + L_2(t) \int_0^{\alpha(t)} e_k(s)\mathrm{d}s + K(t)e_k(\beta(t)). \end{aligned} \tag{2.11}$$

Then, after integration by parts of the first term on the right-hand side, we find

$$\begin{aligned} e_{k+1}(t) \leq{}& L_1(t)\mathrm{e}^{L(t)} \int_0^t \mathrm{e}^{-L(s)} \left\{ (L_2(s)/L_1(s))' \int_0^{\alpha(s)} e_k(z)\mathrm{d}z + (L_2(s)/L_1(s))e_k(\alpha(s))\alpha'(s) \right\} \mathrm{d}s \\ &+ L_1(t)\mathrm{e}^{L(t)} \int_0^t \mathrm{e}^{-L(s)} K(s)e_k(\beta(s))\mathrm{d}s + K(t)e_k(\beta(t)) \\ \leq{}& L_1(t)\mathrm{e}^{L(t)} \int_0^t \frac{\bar{L}_2\alpha'(s)\mathrm{e}^{-L(s)}e_k(\alpha(s))}{\bar{L}_1}\mathrm{d}s + L_1(t)\mathrm{e}^{L(t)} \int_0^t \mathrm{e}^{-L(s)} K(s)e_k(\beta(s))\mathrm{d}s + K(t)e_k(\beta(t)), \end{aligned}$$

i.e.,

$$e_{k+1}(t) \leq h(t) \int_0^t g_1(s)e_k(\alpha(s))\mathrm{d}s + h(t) \int_0^t g_2(s)e_k(\beta(s))\mathrm{d}s + K(t)e_k(\beta(t)), \tag{2.12}$$

where $h(t) = L_1(t)\mathrm{e}^{L(t)}$, $g_1(s) = \frac{\bar{L}_2\alpha'(s)\mathrm{e}^{-L(s)}}{\bar{L}_1}$ and $g_2(s) = K(s)\mathrm{e}^{-L(s)}$. And it is easy to verify that the function $h(t)$ is nonnegative nondecreasing.

Define

$$\begin{cases} \beta_0(t) = t, \beta_{k+1}(t) = \beta_k(\beta(t)), \\ \bar{\beta}(t) = \max\{\alpha(t), \beta(t)\}, \\ \bar{\beta}_0(t) = t, \bar{\beta}_{k+1}(t) = \bar{\beta}_k(\bar{\beta}(t)), \end{cases} \tag{2.13}$$

$$\begin{cases} \pi_0(t) = 1, \\ \pi_k(t) = K(t)K(\beta_1(t)) \cdots K(\beta_{k-1}(t)), \end{cases} \tag{2.14}$$

$$\left\{ \psi_{k+1}(t) = h(t) \int_0^t g_1(s)\psi_k(\alpha(s))\mathrm{d}s + h(t) \int_0^t g_2(s)\psi_k(\beta(s))\mathrm{d}s, \ \psi_0(t) = 1. \right. \tag{2.15}$$

Then, by (2.12) we have the following error estimate.

Theorem 2.1. *Under Condition 1, the error of the WR method (2.5) satisfies the following estimate*

$$e_k(t) \leq e_0(\bar{\beta}_k(t)) \sum_{i=0}^{k} \binom{k}{i} \psi_i(t)\pi_{k-i}(t). \tag{2.16}$$

Proof. We prove this conclusion by an induction method. For $k = 0$, it is obvious that inequality (2.16) holds. Now, assume that (2.16) holds for an arbitrarily fixed k. By (2.12) and the monotonicity properties of $e_0(t)$, $\bar{\beta}_{k+1}(t)$ and

$K(t)$ with respect to t it follows that

$$\begin{aligned} e_{k+1}(t) \le\ & h(t)\int_0^t g_1(s)e_0(\bar{\beta}_{k+1}(s))\sum_{i=0}^{k}\binom{k}{i}\pi_{k-i}(\alpha(s))\psi_i(\alpha(s))\mathrm{d}s + h(t) \\ & \times \int_0^t g_2(s)e_0(\bar{\beta}_{k+1}(s))\sum_{i=0}^{k}\binom{k}{i}\pi_{k-i}(\beta(s))\psi_i(\beta(s))\mathrm{d}s + e_0(\bar{\beta}_{k+1}(t))K(t) \\ & \times \sum_{i=0}^{k}\binom{k}{i}\pi_{k-i}(\beta(t))\psi_i(\beta(t)) \\ \le\ & e_0(\bar{\beta}_{k+1}(t))\left\{\sum_{i=0}^{k}\binom{k}{i}\pi_{k-i}(t)\left[h(t)\int_0^t g_1(s)\psi_i(\alpha(s))\mathrm{d}s + h(t)\int_0^t g_2(s)\psi_i(\beta(s))\mathrm{d}s\right]\right. \\ & \left. +\ K(t)\sum_{i=0}^{k}\binom{k}{i}\pi_{k-i}(\beta(t))\psi_i(\beta(t))\right\} \\ \le\ & e_0(\bar{\beta}_{k+1}(t))\left[\sum_{i=0}^{k}\binom{k}{i}\pi_{k-i}(t)\psi_{i+1}(t) + \sum_{i=0}^{k}\binom{k}{i}\pi_{k+1-i}(t)\psi_i(t)\right] \\ =\ & e_0(\bar{\beta}_{k+1}(t))\sum_{i=0}^{k+1}\binom{k+1}{i}\pi_{k+1-i}(t)\psi_i(t). \end{aligned}$$

This implies that (2.16) holds for k replaced by $k+1$ and according to the induction rule we end the proof. □

Lemma 2.2. *Let $0 < \epsilon < 1$ and $N \ge 1$ be an integer. Then $\lim_{n\to+\infty}\frac{n!}{(n-N)!}\epsilon^{n-N} = 0$.*

Proof. Let $p = \frac{1}{\epsilon}(> 1)$. Then we have $\lim_{n\to+\infty}\frac{n!}{(n-N)!}\epsilon^{n-N} \le \lim_{n\to+\infty} p^N\frac{n^N}{p^n}$. By the principle of L'Hospital it follows that $\lim_{n\to+\infty} p^N\frac{n^N}{p^n} = \lim_{n\to+\infty}\frac{p^N N!}{(\ln p)^N p^n} = 0$. □

Let

$$M(t) = \max\left\{\sup_{0\le s\le t} g_1(s),\ \sup_{0\le s\le t} g_2(s)\right\}, \quad M = \sup_{0\le t\le T} h(t)M(t). \tag{2.17}$$

It is clear that $M < \infty$. By (2.15) and (2.17) we have the following result.

Lemma 2.3. *The function $\psi_k(t)$ defined by (2.15) satisfies*

$$\psi_k(t) \le \frac{(2Mt)^k}{k!}. \tag{2.18}$$

Proof. Obviously, for $k = 0$ the inequality $\psi_0(t) \le 1$ holds. Assume that inequality (2.18) holds for k. Since $\alpha(t) \le t$, $\beta(t) \le t$ for $t \in I$, with (2.15) we have

$$\psi_{k+1}(t) \le M\int_0^t \{\psi_k(\alpha(s)) + \psi_k(\beta(s))\}\mathrm{d}s \le \frac{M(2M)^k}{k!}\int_0^t 2s^k\mathrm{d}s = \frac{(2Mt)^{k+1}}{(k+1)!}.$$

This implies that (2.18) holds for k replaced by $k+1$ and according to the rule of induction we end the proof. □

Therefore, by Lemma 2.3 we can rewrite the error estimate (2.16) as

$$e_k(t) \le e_0(\bar{\beta}_k(t))\sum_{i=0}^{k}\binom{k}{i}\frac{(2tM)^i}{i!}\pi_{k-i}(t). \tag{2.19}$$

Lemma 2.4. *Under Condition 1, we have the following results:*

(1) *the sequence* $\beta_k (k = 0, , 1, \ldots)$ *defined by (2.13) satisfies* $\beta_{k+1}(t) < \beta_k(t)$ *for* $t \in (0, T]$*;*
(2) *for arbitrary* $t \in I$

$$\lim_{k\to+\infty} \beta_k(t) = 0.$$

Proof. It is easy to verify that the sequence β_k satisfies $\beta_{k+1}(t) = \beta(\beta_k(t))$ for $t \in I$ and $k = 0, 1, \ldots$. This relation coupled with $\beta(t) < t$ gives the first statement.

Therefore, it is well known that for arbitrary $t \in (0, T]$ the limitation of $\beta_k(t)$, say $\beta_*(t)$, must exist. And thus it follows that

$$\beta_*(t) = \lim_{k\to+\infty} \beta_{k+1}(t) = \lim_{k\to+\infty} \beta(\beta_k(t)) = \beta(\beta_*(t)),$$

which, combined with (3.2), gives $\beta_*(t) = 0$. Considering $\beta_k(0) = 0$, we have the second statement. □

By (2.19) and the following condition of the functions $K_1(t)$ and $K_2(t)$, we can obtain the convergence of the WR method (2.5) as Theorem 2.2.

Condition 2. Assume that, for $t \in I$ the continuous functions $K_1(t)$ and $K_2(t)$ given in Condition 1 satisfy $K_1(0) + K_2(0) < 1$.

Under this condition, by (2.10) we have

$$K(0) = \frac{K_2(0)}{1 - K_1(0)} < 1. \tag{2.20}$$

Theorem 2.2. *Under Conditions 1 and 2, the WR method (2.5) converges uniformly in interval* $[0, T]$ *and has the error estimate (2.19).*

Proof. Since $e_0(t) < \infty$ for $t \in I$ and the functions $K(t)$ and $\beta(t)$ are both nondecreasing, it follows that it is sufficient to prove

$$\lim_{k\to+\infty} \sum_{i=0}^{k} \binom{k}{i} \frac{(2TM)^i}{i!} \pi_{k-i}(T) = 0.$$

In fact, with $K(0) < 1$ we know that for a fixed constant r_0 satisfying $K(0) \le r_0 < 1$, there exists some σ such that $K(t) < r_0$ for $t \in [0, \sigma]$, since the function $K(t)$ is continuous in interval I. And by the second result of Lemma 2.4, there exists some integer k_0 such that $\beta_k(T) \le \sigma$ holds for $k > k_0$, and this implies

$$K(\beta_k(T)) \le r_0 < 1$$

for $k > k_0$.

And thus we have

$$\pi_{k+1}(T) = \pi_k(T)K(\beta_k(T)) \le \pi_k(T) \quad \text{for } k > k_0$$

and

$$0 \le \lim_{k\to+\infty} \pi_k(T) \le \lim_{k\to+\infty} \{K(T)K(\beta(T)) \cdots K(\beta_{k_0}(T)) {r_0}^{k-k_0-1}\} = 0.$$

By these relations we know that there exists some integer $k_1 > 0$ such that $\pi_k(T) < 1$ for $k > k_1$.

Let

$$k^* = \max\{k_0, k_1\}, r_1 = (\pi_{k^*}(T))^{\frac{1}{k^*}} \quad \text{and} \quad r = \max\{r_0, r_1\},$$

and it is clear that $0 < r < 1$. Therefore, we have

$$\pi_k(T) \le r^k \tag{2.21}$$

for $k \geq k^*$.

Since $0 < r < 1$, there exists some s satisfying $0 < s < 1$ and $r + s < 1$. Consider

$$\lim_{k\to\infty} \frac{1}{\sqrt[k]{k!}} \leq \lim_{k\to\infty} \frac{1}{\sqrt[k]{([\frac{k}{2}]-1)![\frac{k}{2}]^{[\frac{k}{2}]}}} \leq \lim_{k\to\infty} \frac{1}{[\frac{k}{2}]^{\frac{1}{2}(1-\frac{1}{k})}} = 0,$$

where $[X]$ stands for the integer part of X, and thus for $0 < s < 1$ it is clear that there exists some integer $\hat{k}$ such that $\frac{2MT}{(k!)^{1/k}} \leq s$ and $\frac{2MT}{k} \leq s$ for $k \geq \hat{k}$, and this implies

$$\frac{(2MT)^k}{k!} \leq s^k \tag{2.22}$$

for $k \geq \hat{k}$.

Let

$$\hat{M} = \max_{0\leq i\leq \hat{k}-1} \frac{(2MT)^i}{i!} \quad \text{and} \quad M^* = \max_{0\leq i\leq k^*-1} \pi_i(T),$$

and by Lemma 2.2, (2.21) and (2.22) we have

$$\begin{aligned}
\lim_{k\to+\infty} \sum_{i=0}^{k} \binom{k}{i} \frac{(2TM)^i}{i!}\pi_{k-i}(T) &\leq \lim_{k\to+\infty} \sum_{i=0}^{\hat{k}-1} \binom{k}{i} \frac{(2TM)^i}{i!}\pi_{k-i}(T) \\
&\quad + \lim_{k\to+\infty} \sum_{i=\hat{k}}^{k-k^*} \binom{k}{i} s^i r^{k-i} + \lim_{k\to+\infty} \sum_{i=k-k^*+1}^{k} \binom{k}{i} \frac{(2TM)^i}{i!}\pi_{k-i}(T) \\
&\leq \hat{k}\hat{M} \lim_{k\to+\infty} \frac{k! r^{k-\hat{k}+1}}{(k-\hat{k}+1)!} + \lim_{k\to+\infty} (r+s)^k + k^* M^* \lim_{k\to+\infty} \frac{k! s^{k-k^*+1}}{(k-k^*+1)!} \\
&= 0.
\end{aligned}$$

Now, we have completed the proof. □

Remark 2.1. In [6], the convergence conditions are $\beta(t) \leq t$ and $\max_{0\leq t\leq T}(K_1(t)+K_2(t)) < 1$. Our new conditions are $\beta(t) < t$ and $K_1(0) + K_2(0) < 1$. Therefore, in the case $\beta(t) < t$, our result is sharper.

To finish this section, we give some consideration on the case $\beta(t) = t$ on some time points $t = t_i$, $0 \leq i \leq p$. See Fig. 1 as an example. In this case, we assume that the functions $K_1(t)$ and $K_2(t)$ satisfy

$$K_1(t_i) + K_2(t_i) < 1, \quad i = 0, 1, \ldots, p. \tag{2.23}$$

Then, if we execute the WR method (2.5) on these subintervals $[t_0, t_1], [t_1, t_2], \ldots, [t_{p-1}, t_p], [t_p, T]$ one-by-one, by Theorem 2.2 it is obvious that the WR method (2.5) is convergent in each subinterval. Therefore, as we have indicated at the beginning of this section that the convergence of the WR method (2.5) is equivalent to the WR method (1.2), we have the following result.

Corollary 2.1. *If the function $\beta(t)$ given in Condition 1 satisfies $\beta(t) \leq t$ and on the time points t_i $(0 \leq i \leq p)$ that $\beta(t_i) = t_i$ inequality (2.23) is satisfied, the WR method (1.2) converges uniformly in each subinterval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, p-1$.*

3. Superlinear convergence for special cases

For convenience, let $\sigma_k(t) = \sum_{i=0}^{k} \binom{k}{i} \frac{(2tM)^i}{i!}\pi_{k-i}(t)$ and $e_0 = \sup_{0\leq t\leq T} e_0(t)$. Then we can rewrite (2.19) as

$$e_k(t) \leq e_0\sigma_k(t). \tag{3.1}$$

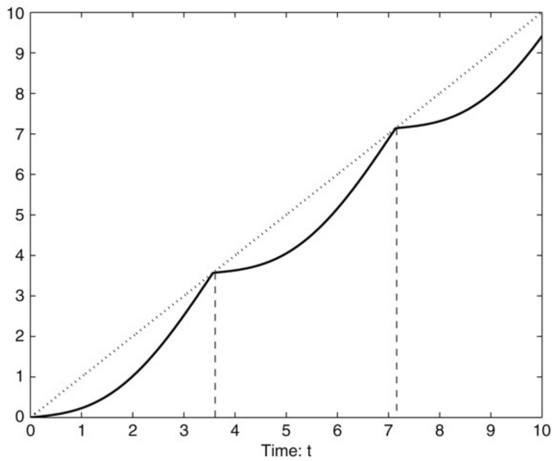

Fig. 1. Sketch of $\beta(t)$.

In this section, we consider two special cases of the function $K_2(t)$ in Condition 1 to show that the convergence of z_k and y_k to y' and y is superlinear, respectively. For this, we will prove that

$$\lim_{k\to+\infty} \frac{\sigma_{k+1}(t)}{\sigma_k(t)} = 0 \tag{3.2}$$

holds for $[0, T]$.

Case A Assume that the function $K_2(t)$ satisfies the following condition.

Condition 3. The function $K_2(t)$ given in Condition 1 satisfies $K_2(0) = 0$.

Under this condition, we have the following theorem.

Theorem 3.1. *Under Conditions 1 and 3, the convergence of $z_k(t)$ and $y_k(t)$ to $y'(t)$ and $y(t)$ is superlinear for $t \in [0, T]$.*

Proof. Define

$$\sigma_{k,\mu}(t) = \sum_{i=\mu}^{k} \frac{P(k,i)}{i!} \tag{3.3}$$

and

$$\tilde{\sigma}_{k,\mu}(t) = \sum_{i=\mu}^{k} \frac{P(k,i)}{(i+1)!}, \tag{3.4}$$

where

$$P(k,i) = \binom{k}{i} (2Mt)^i \pi_{k-i}(t). \tag{3.5}$$

Since $K_2(0) = 0$, from (2.10) we know

$$K(0) = 0.$$

By the right-hand side of (2.19), routine calculations yield

$$\frac{\sigma_{k+1}(t)}{\sigma_k(t)} = 2Mt\frac{\tilde{\sigma}_{k,0}(t)}{\sigma_{k,0}(t)} + \frac{\sum_{i=0}^{k} \frac{P(k,i)K(\beta_{k-i}(t))}{i!}}{\sigma_{k,0}(t)}. \tag{3.6}$$

By (3.6), it is sufficient to prove

$$\lim_{k\to+\infty} 2Mt\frac{\tilde{\sigma}_{k,0}(t)}{\sigma_{k,0}(t)} = 0, \tag{3.7}$$

and

$$\lim_{k\to+\infty} \frac{\sum_{i=0}^{k} \frac{P(k,i)K(\beta_{k-i}(t))}{i!}}{\sigma_{k,0}(t)} = 0 \tag{3.8}$$

for $t \in [0, T]$.

Since the definition of $P(k,i)$ given in (3.5) is essentially different from the one given in [11], we cannot obtain (3.7) by the same proof of Theorem 2 in [11]. Our proof for (3.7) is as follows.

Observe that

$$\begin{aligned}\lim_{k\to+\infty}\sup \frac{2MtP(k,j)/j!}{\sigma_{k,0}} &\le \lim_{k\to+\infty}\sup \frac{2MtP(k,j)/j!}{P(k,j+1)/(j+1)!}\\ &= \lim_{k\to+\infty}\sup \frac{K(\beta_{k-j-1}(t))(j+1)^2}{k-j}\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\lim_{k\to+\infty}\sup \frac{2MtP(k,j)/(j+1)!}{\tilde{\sigma}_{k,0}} &\le \lim_{k\to+\infty}\sup \frac{2MtP(k,j)/(j+1)!}{P(k,j+1)/(j+2)!}\\ &= \lim_{k\to+\infty}\sup \frac{K(\beta_{k-j-1}(t))(j+1)(j+2)}{k-j}\\ &= 0\end{aligned}$$

hold for every fixed integer $j \ge 0$. And this gives

$$\lim_{k\to+\infty}\sup \frac{2Mt\sum_{i=0}^{k^*-1}\frac{P(k,j)}{j!}}{\sigma_{k,0}(t)} = 0$$

and

$$\lim_{k\to+\infty}\sup \frac{2Mt\sum_{i=0}^{k^*-1}\frac{P(k,j)}{(j+1)!}}{\tilde{\sigma}_{k,0}(t)} = 0$$

for every fixed integer $k^* \ge 0$. Therefore

$$\begin{aligned}\lim_{k\to+\infty}\sup \frac{2Mt\tilde{\sigma}_{k,k^*}(t)}{\sigma_{k,k^*}(t)} &= \lim_{k\to+\infty}\sup \frac{2Mt\tilde{\sigma}_{k,0}(t)\left(2Mt - \sum_{j=0}^{k^*-1}\frac{2MtP(k,j)}{(j+1)!}/\tilde{\sigma}_{k,0}(t)\right)}{\sigma_{k,0}(t)\left(2Mt - \sum_{j=0}^{k^*-1}\frac{2MtP(k,j)}{j!}/\sigma_{k,0}(t)\right)}\\ &= \lim_{k\to+\infty}\sup \frac{2Mt\tilde{\sigma}_{k,0}(t)}{\sigma_{k,0}(t)}\end{aligned}$$

holds for every fixed integer $k^* \ge 0$. Next, Obverse that

$$\frac{P(k,j)}{(j+1)!} \le \frac{1}{k^*}\frac{P(k,j)}{j!}$$

holds for $j \geq k^*$. By these relations it follows that

$$\lim_{k\to+\infty}\sup\frac{2Mt\tilde{\sigma}_{k,0}(t)}{\sigma_{k,0}(t)}=\lim_{k\to+\infty}\sup\frac{2Mt\tilde{\sigma}_{k,k^*}(t)}{\sigma_{k,k^*}(t)}\leq\lim_{k\to+\infty}\sup\frac{\frac{2Mt\sigma_{k,k^*}(t)}{k^*}}{\sigma_{k,k^*}(t)}=\frac{2MT}{k^*}$$

holds for every fixed integer $k^* \geq 0$. Passing with k^* to $+\infty$ we get (3.7) and the first part of our proof is finished.

Next, we prove (3.8). For k sufficiently large, observe that

$$\begin{aligned}\frac{\frac{P(k,k-j)K(\beta_j(t))}{(k-j)!}}{\sigma_{k,0}(t)} &\leq \frac{\frac{P(k,k-j)K(\beta_j(t))}{(k-j)!}}{P(k,k-j-1)/(k-j-1)!}\\ &= \frac{j+1}{k-j}\frac{\frac{(2Mt)^{k-j}\pi_j(t)K(\beta_j(t))}{(k-j)!}}{\frac{(2Mt)^{k-j-1}\pi_{j+1}(t)}{(k-j-1)!}}\\ &= 2Mt\frac{j+1}{(k-j)^2}\end{aligned}$$

for every fixed integer $j \geq 0$ and $t \in [0, T]$. And thus for k sufficiently large we find that inequality

$$\frac{\sum_{i=k-k_0+1}^{k}\frac{P(k,i)K(\beta_{k-i}(t))}{i!}}{\sigma_{k,0}(t)}\leq 2Mt\sum_{i=0}^{k_0-1}\frac{i+1}{(k-i)^2}$$

holds for every fixed integer $k_0 \geq 0$ and $t \in [0, T]$.

By those relations and the first result of Lemma 2.4, it follows that

$$\begin{aligned}\lim_{k\to+\infty}\frac{\sum_{i=0}^{k}\frac{P(k,i)K(\beta_{k-i}(t))}{i!}}{\sigma_{k,0}(t)} &= \lim_{k\to+\infty}\frac{\sum_{i=0}^{k-k_0}\frac{P(k,i)K(\beta_{k-i}(t))}{i!}+\sum_{i=k-k_0+1}^{k}\frac{P(k,i)K(\beta_{k-i}(t))}{i!}}{\sigma_{k,0}(t)}\\ &\leq K(\beta_{k_0}(t))+2Mt\lim_{k\to+\infty}\sum_{i=0}^{k_0-1}\frac{i+1}{(k-i)^2}\\ &= K(\beta_{k_0}(t))\end{aligned}$$

holds for every fixed integer $k_0 \geq 0$ and $t \in [0, T]$. Passing with k_0 to infinity we get

$$\lim_{k\to+\infty}\frac{\sum_{i=0}^{k}\frac{P(k,i)K(\beta_{k-i}(t))}{i!}}{\sigma_{k,0}(t)}=0$$

for $t \in [0, T]$, since $K(0) = 0$. $\square$

Remark 3.1. The result given in paper [11] indicates that if $K_2(t) \equiv 0$, the convergence of $z_k(t)$ and $y_k(t)$ to $y'(t)$ and $y(t)$ is superlinear for $t \in [0, T]$. Here we further prove that the condition $K_2(0) = 0$ can guarantee the superlinear convergence.

Case B Next, we assume that the function $K_2(t)$ satisfies some more special condition as follows, and under this condition we can obtain (3.2) in brief.

Condition 4. There exists some θ with $0 < \theta \leq T$ such that the function $K_2(t)$ given in Condition 1 satisfies $K_2(t) = 0$ for $t \in [0, \theta]$.

Under this condition, we know that $K(t) = K_2(t) = 0$ for $t \in [0, \theta]$. And thus by Lemma 2.4, we know that there exists some integer k_1 such that $\beta_k(T) \leq \theta$, and this implies that $K(\beta_k(t)) = 0$ for $k \geq k_1$ and $t \in I$. Therefore

$$\sigma_k(t)=\sum_{i=k-k_1}^{k}\binom{k}{i}\frac{(2Mt)^i}{i!}\pi_{k-i}(t)$$

and

$$\begin{aligned}\lim_{k\to+\infty}\frac{\sigma_{k+1}(t)}{\sigma_k(t)} &\le \lim_{k\to+\infty}\sum_{i=0}^{k_1}\frac{\binom{k+1}{i}\pi_i(t)(2Mt)^{k+1-i}/(k+1-i)!}{\binom{k}{i}\pi_i(t)(2Mt)^{k-i}/(k-i)!}\\ &= 2Mt\lim_{k\to+\infty}\sum_{i=0}^{k_1}\frac{k+1}{(k+1-i)^2}\\ &= 0.\end{aligned}$$

Thus, we obtain the following result.

Theorem 3.2. *Under Conditions 1 and 4, the convergence of $z_k(t)$ and $y_k(t)$ to $y'(t)$ and $y(t)$ is superlinear for $t \in [0, T]$.*

4. Numerical results

In this section, we apply the WR method (1.2) to two test problems to verify the convergence and superlinear convergence results obtained in Sections 2 and 3, respectively. By the results of Sections 2 and 3, we know that the convergence of the WR method (1.2) is mainly influenced by the functions $K_1(t)$ and $K_2(t)$. Therefore, we will simply apply the WR method (1.2) to the test system

$$\begin{cases} y_1'(t) = y_1(t) + y_2(t) + y_1(\alpha t) + B_1(t)y_1'(\beta t) + p_1(t),\\ y_2'(t) = y_2(t) + y_3(t) + y_2(\alpha t) + B_2(t)y_2'(\beta t) + p_2(t),\\ y_3'(t) = y_3(t) + y_1(t) + y_3(\alpha t) + B_3(t)y_3'(\beta t) + p_3(t),\end{cases} \tag{4.1}$$

for $t \in [0, 2]$, where

$$\begin{cases} p_1(t) = 1 - (2+\alpha)t - B_1(t),\\ p_2(t) = 1 - (1+\alpha)t - t^2 - B_2(t),\\ p_3(t) = (1 - 2\beta B_3(t))t - (1+\alpha^2)t^2,\end{cases}$$

and

$$0 < \alpha < 1, \qquad 0 < \beta < 1.$$

Initial conditions:

$$y_1(0) = y_2(0) = y_3(0) = 0.$$

Exact solution:

$$y_1(t) = t, \qquad y_2(t) = t, \qquad y_3(t) = t^2.$$

Let

$$y(t) = (y_1(t), y_2(t), y_3(t))^T, \qquad p(t) = (p_1(t), p_2(t), p_3(t))^T,$$

and then we rewrite (4.1) compactly as

$$\begin{cases} y'(t) = Ay(t) + y(\alpha t) + B(t)y'(\beta t) + p(t), & t \in [0, 2],\\ y(0) = 0, & t = 0,\end{cases} \tag{4.2}$$

where $A = \begin{pmatrix} 110\\ 011\\ 101\end{pmatrix}$ and $B(t) = \begin{pmatrix} B_1(t) & & \\ & B_2(t) & \\ & & B_3(t)\end{pmatrix}$.

With a given initial function $y_0(t)$, we successively solve the equations

$$\begin{cases} y_{k+1}'(t) = Ay_{k+1}(t) + y_k(\alpha t) + B(t)y_k'(\beta t) + p(t), & t \in [0, 2],\\ y_{k+1}(0) = 0, & t = 0.\end{cases} \tag{4.3}$$

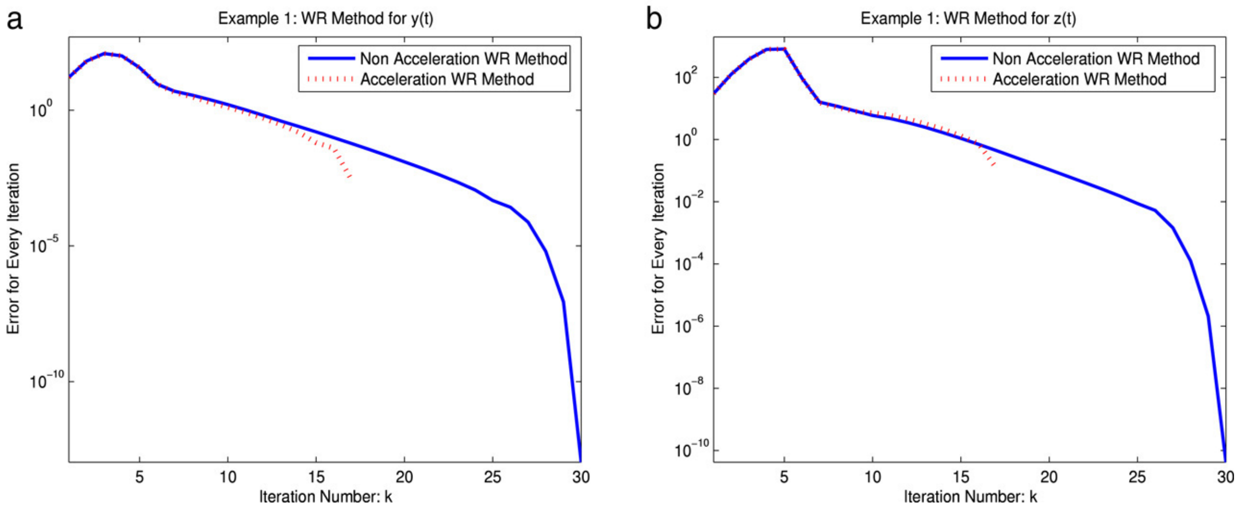

Fig. 2. Convergence rate of the WR method (4.3) for Example 1: on the left for $y(t)$, on the right for $z(t)$ (i.e., $y'(t)$).

Applying Condition 1 to the WR method (4.3), we have

$$K_1(t) = 0, \qquad K_2(t) = \|B(t)\|_\infty,$$

here and below $\|\cdot\|_\infty$ stands for the maximum norm.

In our implementation of the WR method (4.3), we employ the Backward Differential Formula (BDF) method of order 5 with step-size $h = 0.001$. Let $y_k^h(t_i)(i = 0, 1, \ldots, N)$ be the sequence generated by the discrete-time WR method of (4.3) and $y^h(t_i)$ be the sequence generated by directly applying the underlying BDF method to system (4.2), where $N = [\frac{2}{h}] = 2000$. After calculating $y_k^h(t_i)$ and $y^h(t_i)$, we set $z_k^h(t_i) = \frac{y_k^h(t_{i+1}) - y_k^h(t_i)}{h}$ and $z^h(t_i) = \frac{y^h(t_{i+1}) - y^h(t_i)}{h}$ for $i = 0, 1, \ldots, N-1$.

To carry out these calculations, the underlying BDF method required not only the values of the previous solution y_k^h at grid points t_i but also its values at points αt_i and the derivatives of y_k^h at βt_i. We use Lagrange interpolating polynomial of order 4 and its derivative to evaluate these lacking values.

It is clear by (2.13) that $\bar{\beta}(t) < t$ for $t \in (0, 2]$ in the test system (4.2). And thus, by observing the term $e_0(\bar{\beta}_k(t))$ contained in the error estimate (2.19), we know that if the initial function $y_0(t)$ is very close to the true solution $y(t)$ in some subinterval $[0, \delta]$ of I (i.e., $[0, \delta] \subseteq [0, T]$), the convergence of the WR method (4.3) on the whole interval I will be faster. Therefore, with an arbitrary initial function $y_0(t)$, we first calculate the numerical values $y^h(t_i)$ in a small interval, say $[0, 0.05]$, by the underlying BDF method with step-size $h = 0.001$, and then we use $\bar{y}_0(t_i) = \begin{cases} y^h(t_i), & t_i \in [0, 0.05], \\ y_0(t_i), & t_i \in [0.05, 2], \end{cases}$ as the new initial function. By this new initial function $\bar{y}_0(t)$, we execute the computation in interval $[0.05, 2]$ with the same step size. For convenience, we call this strategy the initial interval acceleration.

In our test examples, we set $\alpha = 0.8$ and $\beta = 0.795$.

Example 1. Set $B(t) = \begin{pmatrix} 3(\cos(t) - 1) + 0.35 & & \\ & \sin(4t^2) + 0.5 & \\ & & 4(\cos(2t^2) - 1) + 0.75 \end{pmatrix}$ in (4.2) and choose the initial function $y_0(t) = (I - B(0))^{-1}((A + I)y(0) + p(0))t + y(0)$ to satisfy the consistent conditions $y_0(0) = y(0)$ and $y_0'(0) = y'(0)$. It is obvious that $K_1(0) + K_2(0) = 0.75$ in this example, and by Theorem 2.2 we know that the WR method (4.3) will be convergent.

Fig. 2(a) and (b) are the error figures of $\max_{0 \le i \le N} \|y_k^h(t_i) - y^h(t_i)\|_\infty$ and $\max_{0 \le i \le N} \|z_k^h(t_i) - z^h(t_i)\|_\infty$ for 30 iterations with and without initial interval acceleration strategy.

Example 2. We set $B(t) = \begin{pmatrix} 3(\cos(t) - 1) & & \\ & \sin(4t^2) & \\ & & 4(\cos(2t^2) - 1) \end{pmatrix}$ in (4.2) and choose the initial function $y_0(t) = (I - B(0))^{-1}((A + I)y(0) + p(0))t + y(0)$ to guarantee the consistent conditions. Since $K_2(0) = 0$ in this example, by Theorem 3.1 we know that the convergence of $y_k(t)$ and $z_k(t)$ to $y(t)$ and $y'(t)$ will be superlinear.

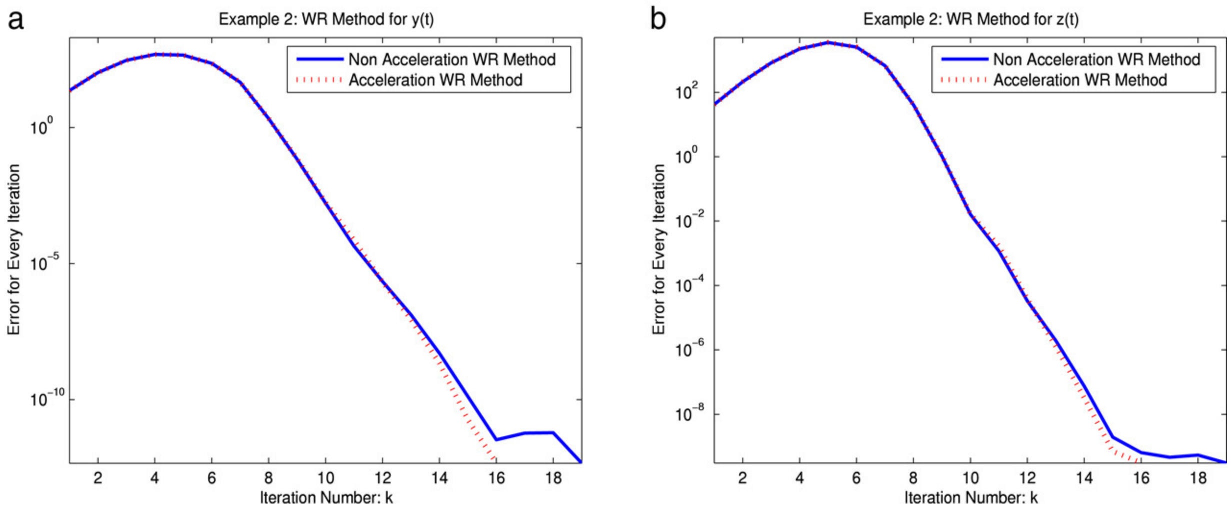

Fig. 3. Convergence rate of the WR method (4.3) for Example 2: on the left for $y(t)$, on the right for $z(t)$ (i.e., $y'(t)$).

Table 1
Example 1

Non-acceleration	Acceleration
$\max_{0\le i\le N} \|y_{30}^h(t_i) - y^h(t_i)\|_\infty = 1.1 \times 10^{-13}$	$\max_{0\le i\le N} \|y_{18}^h(t_i) - y^h(t_i)\|_\infty = 0$
$\max_{0\le i\le N} \|z_{30}^h(t_i) - z^h(t_i)\|_\infty = 4.2 \times 10^{-11}$	$\max_{0\le i\le N} \|z_{18}^h(t_i) - z^h(t_i)\|_\infty = 0$

Table 2
Example 2

Non-acceleration	Acceleration
$\max_{0\le i\le N} \|y_{19}^h(t_i) - y^h(t_i)\|_\infty = 4.6 \times 10^{-13}$	$\max_{0\le i\le N} \|y_{17}^h(t_i) - y^h(t_i)\|_\infty = 0$
$\max_{0\le i\le N} \|z_{19}^h(t_i) - z^h(t_i)\|_\infty = 3.0 \times 10^{-10}$	$\max_{0\le i\le N} \|z_{17}^h(t_i) - z^h(t_i)\|_\infty = 0$

Fig. 3(a) and (b) are the error figures of $\max_{0\le i\le N} \|y_k^h(t_i) - y^h(t_i)\|_\infty$ and $\max_{0\le i\le N} \|z_k^h(t_i) - z^h(t_i)\|_\infty$ for 19 iterations with and without initial interval acceleration strategy.

In Tables 1 and 2, we list $\max_{0\le i\le N} \|y_k^h(t_i) - y^h(t_i)\|_\infty$ and $\max_{0\le i\le N} \|z_k^h(t_i) - z^h(t_i)\|_\infty$ for those two examples after $k = 30, 19$ non-acceleration WR method, and $k = 18, 17$ acceleration WR method, respectively.

Now, it is clear from Figs. 2 and 3 that, since the function $K(t)$ in these two examples satisfies the condition $K_1(0) + K_2(0) < 1$, the WR method (4.3) is convergent. However, such convergence results cannot be guaranteed by [11] and [6], since $\max_{0\le t\le 2} K_1(t) + K_2(t) > 1$ in both Examples 1 and 2. For the superlinear convergence aspect, from Tables 1 and 2, we find that compared with Example 2, 11 additional iterations with non-acceleration WR method are needed by Example 1 to achieve the error tolerance 10^{-13}.

It is also clear that the initial interval acceleration strategy applied to the discrete-time WR method has essential acceleration potentiality, since only 18 iterations are needed to achieve the true numerical solution for Example 1, and 17 iterations are needed for Example 2.

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